Yiddish of the Day

"Vos makhstu" = וואָס מאַכסטו?

oder = אָדער

"Vos makht a yid" = וואָס מאַכט אַ ייִד?

what is up =

DO THE SETS

Lecture 11 - Inner Product

Recall

- We based our idea of a vector space on $\mathbb{R}^n$

  $\longrightarrow$ but $\mathbb{R}^n$ has more stuff!

  (length , distance , angles)

- Really these come from the same thing!

Def: Let V be $\mathbb{R}$ v.s. Then an inner-product on V is a bilinear form

$$\langle -, - \rangle : V \times V \longrightarrow \mathbb{R}$$

st 1) $\langle x, y \rangle = \langle y, x \rangle \quad \forall x, y \in V$ (symmetric)

2) $\langle x, x \rangle \geq 0$

(and $= 0_{\mathbb{R}}$ iff $x = 0_V$)

("positive definite")

ex) $V = \mathbb{R}^n$ with $\langle \vec{x}, \vec{y} \rangle = \vec{x} \cdot \vec{y} = \sum_{i=1}^{n} x_i y_i$ "dot product"

Def 2: An inner-product on a $\mathbb{C}$ vector space V is a function

$$\langle -, - \rangle : V \times V \to \mathbb{C} \quad \text{st}$$

1) $\langle -, - \rangle$ is linear in first slot

$$\Big(\langle \lambda_1 x_1 + x_2, x_3 \rangle = \lambda_1 \langle x_1, x_3 \rangle + \langle x_2, x_3 \rangle \Big)$$

2) $\langle -, - \rangle$ is anti - linear in second slot

$$\left(\langle x_1, \lambda_1 x_2 + x_3 \rangle = \overline{\lambda_1} \langle x_1, x_2 \rangle + \langle x_1, x_3 \rangle \right)$$

3) $\langle x, y \rangle = \underline{\overline{\langle y, x \rangle}}$

4) $\langle x, x \rangle \geq \underline{0}$

[Q: Why do we know $\langle x, x \rangle \in \underline{\mathbb{R}}$?]

and $= \underline{0}$ iff $x = \underline{0}$

ex) $V = \mathbb{C}^n$ $\quad \langle \vec{\alpha}, \vec{\beta} \rangle = \sum_{i=1}^{n} \overline{\alpha_i} \beta_i$

What does this give us?

Def: i) Let V be either $\mathbb{R}$ or $\mathbb{C}$-vs and let $\langle -, - \rangle$ be an inner-product on V.

Say v_1, v_2 are orthogonal if

$$\langle v_1, v_2 \rangle = 0$$

ii) Say a set $S \subseteq V$ is orthonormal (ON) if $\forall x, y \in S$

$$\langle x, y \rangle = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$$

iii) Define the <u>length</u> of $x \in V$ to be

$$\|x\| = \sqrt{\langle x, x \rangle}$$

ex) $V = \mathbb{R}^n$ ($\mathbb{C}^n$) the standard basis is ON

Prop: Suppose $S = (v_1, \dots, v_k)$ (all these are non-zero) is an <u>orthogonal</u> set. Then S is also <u>linearly independent</u>

Pf) Suppose $\underline{c_1 v_1 + \dots + c_k v_k = 0}$ for $c_1 \dots c_k \in \mathbb{F}$

Apply $\langle -, v_1 \rangle$ to both sides, ie

$$\langle c_1 v_1 + \dots + c_k v_k, v_1 \rangle = \langle 0, v_1 \rangle = 0$$

$$c_1 \langle v_1, v_1 \rangle + c_2 \langle \cancel{v_2}, v_1 \rangle + \dots + c_k \langle \cancel{v_k}, v_1 \rangle$$

$\Rightarrow c_1 = 0$. Repeat with v_i :)

- So <u>orthogonal vectors</u> are nice. They can be even nicer.

<u>Def</u>: Say $B = (v_1 \dots v_n)$ are an <u>orthonormal</u> - <u>basis</u> of V if

1) it's a basis

2) it's an ON set

Prop: Suppose $\mathcal{B} = (v_1 \ldots v_n)$ is an orthonormal - basis of V

Then for any $w \in V$ we have

$$w = \sum_{i=1}^{n} \langle w, v_i \rangle v_i$$

Pf) We know that $w = c_1 v_1 + \ldots + c_n v_n$ for $c_1 \ldots c_n \in \mathbb{F}$

Again, apply $\langle -, v_i \rangle$ to both sides.

Then $\langle w, v_i \rangle = \langle c_1 v_1 + \ldots + c_n v_n, v_i \rangle$

$= c_1 \langle v_1, v_i \rangle + c_2 \langle v_2, v_i \rangle + \ldots + c_n \langle v_n, v_i \rangle$

$$= c_i \langle v_i, v_i \rangle = c_i \qquad \square$$

Thm: V an inner-product space. Then V has an orthonormal-basis

(*) Google "Grahm-Schmidt"

Something else an inner-product lets us do is remove our dependence on choosing a basis.

<u>Def</u>: $W \leq V$. Define the <u>orthogonal compliment</u> to be

$$W^{\perp} = \{ w' \in V \mid \langle w', w \rangle = 0 \ \forall w \in W \}$$

<u>Prop</u>: For $W \leq V$ we have

$$V = \underline{W \oplus W^{\perp}}$$

Moreover $(W^\perp)^\perp = W$

Pf) Suppose $v \in W \cap W^\perp$. Then

$\langle v, v \rangle = 0$

Now let $v \in V$ arbitrary. Write

[Will return to this after class is done]

Let $w_1, \dots, w_k$ be ON basis for V.

Define the map

$P : V \to V$ by $P(v) = \sum_{i=1}^{k} \langle v, w_i \rangle w_i$

Then note that $\operatorname{im} P \subseteq W$ and in fact

$\operatorname{im} P = W$ since $P(w_i) = w_i$ for each basis vector.

Now we claim $\ker P = W^\perp$. Indeed

$\ker P = \{ v \in V \mid \sum \langle v, w_i \rangle w_i = 0 \}$

$= \{ v \in V \mid \langle v, w_i \rangle = 0 \ \forall w_i \text{ (since } w_i \text{ are LI)} \}$

$= \{ v \in V \mid \langle v, w \rangle = 0 \ \forall w \in W \text{ since } w_i \text{ are basis} \}$

$= W^\perp$

Then the rank-nullity thm tells us that

$$\dim V = \dim(\operatorname{im} P) + \dim(\operatorname{Ker}(P))$$
$$= \dim(W) + \dim(W^{\perp})$$

$$\Rightarrow V = W + W^{\perp}$$

What this avoids is having to choose an extension of my basis of W. Ie there is a "canonical" complimentary subspace for W

There is another way to think about this:

V be either a $\mathbb{R}$ or $\mathbb{C}$ inner product space

Then we have a map

$$\varphi : V \longrightarrow V^*$$
$$v \longmapsto \langle -, v \rangle$$

- When V is $\mathbb{R}$-vs this is linear

  V is $\mathbb{C}$-vs this is anti-linear $(\varphi(\lambda_1 v_1 + v_2) = \bar{\lambda}_1 \varphi_1(v_1) + \varphi(v_2))$

<u>Thm</u>: This map $\delta : V \to V^*$ is a <u>bijection</u> when V is finite dimensional.

Pf) <u>Skip</u> lol

This says that any linear functional $\phi \in V^*$ is of the form $\phi(-) = \langle -, v \rangle$ $\exists! \, v$

$\phi(w) := \langle w, v \rangle$

$\Rightarrow$ This gives us a basis free ("canonical") isomorphism

$V \cong V^*$ (when V is $\mathbb{R}$-vs)

This will allow us to identify V with V^* as we did before with V and V^{**}

$\rightsquigarrow$ linear maps on $\underline{\quad V \quad}$ can now be thought of as being on $\underline{\quad V^* \quad}$

$$\left(\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "g"', "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & V \arrow[d, "g", "\wr"'] \\ V^* & V^* \arrow[l, "T^*"'] \end{tikzcd}\right)$$

<u>Def / Thrm</u>: Let V be fd inner-product space and $T : V \to V$, linear. Then $\exists!$ linear map $T^* : V \to V$ st

$$\underline{\langle T(v), w \rangle = \langle v, T^*(w) \rangle} \quad \forall v, w \in V$$

(this map is called the "adjoint" of T)

Pf) $V \xrightarrow{g} V^* \xrightarrow[\text{the dual map}]{T^*} V^* \xrightarrow{g^{-1}} V$

$v \longrightarrow \langle -, v \rangle \longrightarrow \langle T(-), v \rangle \longrightarrow$ the unique v' st

$\langle T(-), v \rangle = \langle -, v' \rangle$

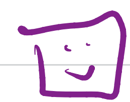
$\phi(w) := \langle T(w), v \rangle$

$\phi(w) = \langle w, v' \rangle$

$\langle Tw, v \rangle = \langle w, T^* v \rangle$

$T^*(v) = v'$

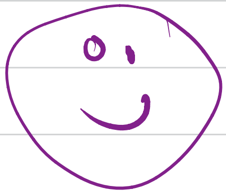
$\langle T(w), v \rangle = \langle w, T^*(v) \rangle$



Prove this is linear!

Thm: $T: V \to V$ linear and $\mathcal{B} = (v_1, \dots, v_n)$ orthonormal

Then $[T^*]_{\mathcal{B}} = \overline{[T]}_{\mathcal{B}}^{tr}$

Pf) HW 

(**Hint**: Let $T(v_i) = \sum_{j=1}^{n} a_{ij} v_j$

Compute $\langle Tv_i, v_j \rangle$)

(if time, do below)

<u>Prop</u>: Let $S, T : V \to V$ linear and $\alpha \in \mathbb{F}$ (either $\mathbb{R}$, $\mathbb{C}$)

Then 1) $(S+T)^* = S^* + T^*$

exc 2) $(\alpha S)^* = \bar{\alpha} S^*$

3) $(S \circ T)^* = T^* \circ S^*$

exc 4) $(S^*)^* = S$

<u>challenge</u> 5) $\det(T^*) = \overline{\det(T)}$

Pf) 1) $\langle (S+T)(v), w \rangle$

$$\shortparallel$$

$\langle S(v)+T(v), w \rangle$

$$\shortparallel$$

$\langle S(v), w \rangle + \langle T(v), w \rangle = \langle v, S^*(w) \rangle + \langle v, T^*(w) \rangle$

$= \langle v, S^*(w) + T^*(w) \rangle$

$= \langle v, (S^* + T^*)(w) \rangle$

$\Rightarrow (S+T)^* = S^* + T^*$

3) $\langle ST(v), w \rangle = \langle T(v), S^* w \rangle$

$= \langle v, T^* S^*(w) \rangle$

$$(ST)^* = T^*S^*$$